

ON GENERALIZATION OF D'AURIZIO-SÁNDOR TRIGONOMETRIC INEQUALITIES WITH A PARAMETER

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Abstract. In this work, we generalize the D'Aurizio-Sándor inequalities ([2, 4]) using an elementary approach. In particular, our approach provides an alternative proof of the D'Aurizio-Sándor inequalities. Moreover, as an immediate consequence of the generalized D'Aurizio-Sándor inequalities, we establish the D'Aurizio-Sándor-type inequalities for hyperbolic functions.

1. Introduction

Based on infinite product expansions and inequalities on series and the Riemann's zeta function, D'Aurizio ([2]) proved the following inequality:

$$\frac{1 - \frac{\cos x}{\cos \frac{x}{2}}}{x^2} < \frac{4}{\pi^2}, \quad (1)$$

where $x \in (0, \pi/2)$. Using an elementary approach, Sándor ([4]) offered an alternative proof of (1) by employing trigonometric inequalities and an auxiliary function. In the same paper, Sándor also provided the converse to (1):

$$\frac{1 - \frac{\cos x}{\cos \frac{x}{2}}}{x^2} > \frac{3}{8}, \quad (2)$$

where $x \in (0, \pi/2)$. In addition, Sándor found the following analogous inequality (4) holds true for the case of sine functions:

THEOREM 1. (D'Aurizio-Sándor inequalities ([2, 4])) *The two double inequalities*

$$\frac{3}{8} < \frac{1 - \frac{\cos x}{\cos \frac{x}{2}}}{x^2} < \frac{4}{\pi^2} \quad (3)$$

and

$$\frac{4}{\pi^2} (2 - \sqrt{2}) < \frac{2 - \frac{\sin x}{\sin \frac{x}{2}}}{x^2} < \frac{1}{4} \quad (4)$$

hold for any $x \in (0, \pi/2)$.

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Throughout this paper, we denote $\frac{1 - \frac{\cos x}{\cos \frac{x}{p}}}{x^2}$ and $\frac{p - \frac{\sin x}{\sin \frac{x}{p}}}{x^2}$ by $f_c(x)$ and $f_s(x)$, respectively:

$$f_p^c(x) = \frac{1 - \frac{\cos x}{\cos \frac{x}{p}}}{x^2}, \quad (5)$$

$$f_p^s(x) = \frac{p - \frac{\sin x}{\sin \frac{x}{p}}}{x^2}. \quad (6)$$

Our aim is to generalize the D'Aurizio-Sándor inequalities for the case of $f_p^c(x)$ and $f_p^s(x)$ as follows:

THEOREM 2. (Generalized D'Aurizio-Sándor inequalities) *Let $0 < x < \pi/2$. Then the two double inequalities*

$$\frac{4}{\pi^2} < \frac{1 - \frac{\cos x}{\cos \frac{x}{p}}}{x^2} < \frac{p^2 - 1}{2p^2} \quad (7)$$

and

$$\frac{4}{\pi^2} \left(p - \csc \left(\frac{\pi}{2p} \right) \right) < \frac{p - \frac{\sin x}{\sin \frac{x}{p}}}{x^2} < \frac{p^2 - 1}{6p} \quad (8)$$

hold for $p = 3, 4, 5, \dots$. In particular, the double inequality (8) remains true when $p = 2$ while the double inequality (7) is reversed when $p = 2$.

The remainder of this paper is organized as follows. Section 2 is devoted to the proof of Theorem 2 and an alternative proof of Theorem 1. In Section 3, we establish analogue of Theorem 2 for hyperbolic functions. As an application of Theorem 2, we apply in Section 4 inequality (8) to the Chebyshev polynomials of the second kind and establish a trigonometric inequality.

2. Proof of the main results

At first we will prove the following lemma. The lemma provides expressions of the higher-order derivative $\frac{d^2}{dx^2}(x^3 \frac{d}{dx} f_p^\Delta(x))$ involving $f_p^\Delta(x)$ ($\Delta = c, s$), which are helpful in proving Theorem 2. We note that the sign of $\frac{d^2}{dx^2}(x^3 \frac{d}{dx} f_p^\Delta(x))$ plays a crucial role in proving Theorem 2.

LEMMA 1. *Let $0 < x < \pi/2$ and $k = 1, 2, 3, \dots$. Then when $p \in \mathbb{R}$ and $p \neq 0$, we have*

(i)

$$\begin{aligned} \frac{d^2}{dx^2} \left(x^3 \frac{d}{dx} f_p^c(x) \right) &= -\frac{x \csc^4 \left(\frac{x}{p} \right)}{8p^3} \left((p+1)^3 \sin \left(x - \frac{3x}{p} \right) + (p-1)^3 \sin \left(x + \frac{3x}{p} \right) \right. \\ &\quad \left. + (3p^3 + 3p^2 - 15p - 23) \sin \left(x - \frac{x}{p} \right) + (3p^3 - 3p^2 - 15p + 23) \sin \left(x + \frac{x}{p} \right) \right); \end{aligned} \quad (9)$$

(ii)

$$\begin{aligned} \frac{d^2}{dx^2} \left(x^3 \frac{d}{dx} f_p^s(x) \right) &= \frac{x \csc^4 \left(\frac{x}{p} \right)}{8p^3} \left((p+1)^3 \sin \left(x - \frac{3x}{p} \right) - (p-1)^3 \sin \left(x + \frac{3x}{p} \right) \right. \\ &\quad \left. + (-3p^3 - 3p^2 + 15p + 23) \sin \left(x - \frac{x}{p} \right) + (3p^3 - 3p^2 - 15p + 23) \sin \left(x + \frac{x}{p} \right) \right). \end{aligned} \quad (10)$$

In particular,

(iii) when $p = 2k$,

$$\frac{d^2}{dx^2} \left(x^3 \frac{d}{dx} f_p^s(x) \right) = -\frac{x}{4k^3} \sum_{j=0}^{k-1} (2j+1)^3 \sin \left(\frac{2j+1}{2k} x \right); \quad (11)$$

(iv) when $p = 2k+1$,

$$\frac{d^2}{dx^2} \left(x^3 \frac{d}{dx} f_p^c(x) \right) = -\frac{16x}{(2k+1)^3} \sum_{j=1}^k j^3 \sin \left(\frac{2j}{2k+1} x \right) (-1)^{j-1}, \quad (12)$$

$$\frac{d^2}{dx^2} \left(x^3 \frac{d}{dx} f_p^s(x) \right) = -\frac{16x}{(2k+1)^3} \sum_{j=1}^k j^3 \sin \left(\frac{2j}{2k+1} x \right). \quad (13)$$

(v) For $\triangle = c, s$ and $p \in \mathbb{R} \setminus \{0\}$,

$$\lim_{x \rightarrow 0} \frac{d}{dx} \left(x^3 \frac{d}{dx} f_p^\triangle(x) \right) = \lim_{x \rightarrow 0} x^3 \frac{d}{dx} f_p^\triangle(x) = 0. \quad (14)$$

Proof. (i), (ii) and (v) follows directly from calculations using elementary Calculus. In particular, trigonometric addition formulas are used in proving (i) and (ii). To prove (11), we claim

$$-\frac{x}{4k^3} \sum_{j=0}^{k-1} (2j+1)^3 \sin \left(\frac{2j+1}{2k} x \right) = -x \frac{d^3}{dx^3} \left(\frac{\sin x}{\sin \left(\frac{x}{2k} \right)} \right). \quad (15)$$

Indeed, we rewrite

$$\frac{1}{4k^3} \sum_{j=0}^{k-1} (2j+1)^3 \sin\left(\frac{2j+1}{2k}x\right) = 2 \frac{d^3}{dx^3} \left(\sum_{j=0}^{k-1} \cos\left(\frac{2j+1}{2k}x\right) \right). \quad (16)$$

On the other hand, making use of Euler's formula $e^{iz} = \cos z + i \sin z$ leads to an alternative expression of the left-hand side of (16):

$$\sum_{j=0}^{k-1} \cos\left(\frac{2j+1}{2k}x\right) = \sum_{j=0}^{k-1} \Re \left\{ e^{i\left(\frac{x}{2k} + \frac{j}{k}\right)} \right\} = \Re \left\{ e^{i\frac{x}{2k}} \sum_{j=0}^{k-1} \left(e^{i\frac{x}{k}} \right)^j \right\} \quad (17)$$

$$= \Re \left\{ e^{i\frac{x}{2k}} \frac{1 - e^{ix}}{1 - e^{i\frac{x}{k}}} \right\} = \Re \left\{ e^{i\frac{x}{2k}} \frac{e^{\frac{ix}{2}} (e^{-\frac{ix}{2}} - e^{\frac{ix}{2}})}{e^{\frac{ix}{k}} (e^{-\frac{ix}{2k}} - e^{\frac{ix}{2k}})} \right\} \quad (18)$$

$$= \Re \left\{ e^{\frac{ix}{2}} \frac{\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2k}\right)} \right\} = \cos\left(\frac{x}{2}\right) \frac{\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2k}\right)} = \frac{\sin x}{2 \sin\left(\frac{x}{2k}\right)}, \quad (19)$$

where $\Re\{z\}$ is the real part of z and $i = \sqrt{-1}$. Now it suffices to show

$$\frac{d^2}{dx^2} \left(x^3 \frac{d}{dx} f_p^s(x) \right) = -x \frac{d^3}{dx^3} \left(\frac{\sin x}{\sin\left(\frac{x}{2k}\right)} \right). \quad (20)$$

Using (10) in (ii), this can be achieved by straightforward calculations. Thus (iii) is true. The proof of (iv) is similar, and we omit the details. We complete the proof of Lemma 1. \square

We provide here an alternative proof of the two double inequalities in Theorem 1.

Proof. [Proof of Theorem 1] To this end, we show that for $x \in (0, \pi/2)$, $f_2^c(x) = \frac{1 - \frac{\cos x}{\cos \frac{x}{2}}}{x^2}$ is strictly increasing while $f_2^s(x) = \frac{2 - \frac{\sin x}{\sin \frac{x}{2}}}{x^2}$ is strictly decreasing. These lead to the desired inequalities since it is easy to see that

$$\lim_{x \rightarrow 0} f_2^c(x) = \frac{3}{8}, \quad \lim_{x \rightarrow \pi/2} f_2^c(x) = \frac{4}{\pi^2}, \quad (21)$$

$$\lim_{x \rightarrow 0} f_2^s(x) = \frac{1}{4}, \quad \lim_{x \rightarrow \pi/2} f_2^s(x) = \frac{4}{\pi^2} (2 - \sqrt{2}). \quad (22)$$

To see $f_2^c(x)$ is strictly increasing, we employ (9) in Lemma 1 to obtain

$$\begin{aligned} \frac{d^2}{dx^2} \left(x^3 \frac{d}{dx} f_2^c(x) \right) &= -\frac{x}{64} \sec^4\left(\frac{x}{2}\right) \left(-44 \sin\left(\frac{x}{2}\right) + 5 \sin\left(\frac{3x}{2}\right) + \sin\left(\frac{5x}{2}\right) \right) \\ &= -\frac{x}{16} \sec^4\left(\frac{x}{2}\right) \sin\left(\frac{x}{2}\right) (\cos x - 2)(\cos x + 5) > 0. \end{aligned} \quad (23)$$

As $\lim_{x \rightarrow 0} \frac{d}{dx} \left(x^3 \frac{d}{dx} f_2^c(x) \right) = 0$, it follows that $\frac{d}{dx} \left(x^3 \frac{d}{dx} f_2^c(x) \right) > 0$. We are led to $x^3 \frac{d}{dx} f_2^c(x) > 0$ or $\frac{d}{dx} f_2^c(x) > 0$ since $\lim_{x \rightarrow 0} \left(x^3 \frac{d}{dx} f_2^c(x) \right) = 0$. This that shows $f_2^c(x)$ is strictly increasing.

By using (11) in Lemma 1, we have

$$\frac{d^2}{dx^2} \left(x^3 \frac{d}{dx} f_2^s(x) \right) = -\frac{x}{4} \sin\left(\frac{x}{2}\right) < 0, \quad (24)$$

from which we infer that $\frac{d}{dx} \left(x^3 \frac{d}{dx} f_2^s(x) \right) < 0$ since $\lim_{x \rightarrow 0} \frac{d}{dx} \left(x^3 \frac{d}{dx} f_2^s(x) \right) = 0$ by (v) of Lemma 1. Then

$$\frac{d}{dx} \left(x^3 \frac{d}{dx} f_2^s(x) \right) < 0 \quad (25)$$

together with the fact $\lim_{x \rightarrow 0} \left(x^3 \frac{d}{dx} f_2^s(x) \right) = 0$ from (v) of Lemma 1 yields $x^3 \frac{d}{dx} f_2^s(x) < 0$ or $\frac{d}{dx} f_2^s(x) < 0$. Thus we have shown that $f_2^s(x)$ is strictly decreasing. This completes the proof of the theorem. \square

We are now in the position to give the proof of Theorem 2.

Proof. [Proof of Theorem 2] The proof of the case when $p = 2$ has been given in Theorem 2. For $p \geq 3$, we prove the desired inequalities by showing that $\frac{d}{dx} f_p^\Delta(x) < 0$ for $\Delta = c, s$. Due to (i) of Lemma 1, we see that $\frac{d^2}{dx^2} \left(x^3 \frac{d}{dx} f_p^c(x) \right) < 0$ for $p \geq 3$. Instead of employing (ii) of Lemma 1, we use (11) and (13) in Lemma 1 to conclude that $\frac{d^2}{dx^2} \left(x^3 \frac{d}{dx} f_p^s(x) \right) < 0$. Thus we have for $\Delta = c, s$,

$$\frac{d^2}{dx^2} \left(x^3 \frac{d}{dx} f_p^\Delta(x) \right) < 0. \quad (26)$$

Because of the first vanishing limit in (v) of Lemma 1, it follows that

$$\frac{d}{dx} \left(x^3 \frac{d}{dx} f_p^\Delta(x) \right) < 0, \quad (27)$$

which, together with the fact that the second limit in (v) of Lemma 1 vanishes, implies that $x^3 \frac{d}{dx} f_p^\Delta(x) < 0$ or $\frac{d}{dx} f_p^\Delta(x) < 0$ for $\Delta = c, s$. It remains to find the following limits:

$$\lim_{x \rightarrow 0} f_p^c(x) = \frac{p^2 - 1}{2p^2}, \quad \lim_{x \rightarrow \pi/2} f_p^c(x) = \frac{4}{\pi^2}, \quad (28)$$

$$\lim_{x \rightarrow 0} f_p^s(x) = \frac{p^2 - 1}{6p}, \quad \lim_{x \rightarrow \pi/2} f_p^s(x) = \frac{4}{\pi^2} \left(p - \csc\left(\frac{\pi}{2p}\right) \right). \quad (29)$$

We immediately have

$$\frac{4}{\pi^2} = \lim_{x \rightarrow \pi/2} f_p^c(x) < \frac{1 - \frac{\cos x}{\cos \frac{x}{p}}}{x^2} < \lim_{x \rightarrow 0} f_p^c(x) = \frac{p^2 - 1}{2p^2} \quad (30)$$

and

$$\frac{4}{\pi^2} \left(p - \csc\left(\frac{\pi}{2p}\right) \right) = \lim_{x \rightarrow \pi/2} f_p^s(x) < \frac{p - \frac{\sin x}{\sin \frac{x}{p}}}{x^2} < \lim_{x \rightarrow 0} f_p^s(x) = \frac{p^2 - 1}{6p}. \quad (31)$$

The proof is completed. \square

3. Generalized D'Aurizio-Sándor inequalities for hyperbolic functions

In this section, we show an analogue of Theorem 2 for the case of hyperbolic functions holds true. Let

$$h_p^c(x) = \frac{1 - \frac{\cosh x}{\cosh \frac{x}{p}}}{x^2}, \quad (32)$$

$$h_p^s(x) = \frac{p - \frac{\sinh x}{\sinh \frac{x}{p}}}{x^2}. \quad (33)$$

Following the same arguments for proving Lemma 1, it can be shown that Lemma 1 with $\cos x$, $\sin x$ and $f_p^\Delta(x)$ ($\Delta = c, s$) replaced by $\cosh x$, $\sinh x$ and $h_p^\Delta(x)$ ($\Delta = c, s$) respectively, remains true. It follows that we can prove $\frac{d}{dx}h_p^\Delta(x) < 0$ for $\Delta = c, s$ as in the proof of Theorem 2. It remains to calculate the following limits:

$$\lim_{x \rightarrow 0} f_p^c(x) = \frac{1 - p^2}{2p^2}, \quad \lim_{x \rightarrow \pi/2} f_p^c(x) = \frac{4}{\pi^2} \left(1 - \cosh\left(\frac{\pi}{2}\right) \operatorname{sech}\left(\frac{\pi}{2p}\right) \right), \quad (34)$$

$$\lim_{x \rightarrow 0} f_p^s(x) = \frac{1 - p^2}{6p}, \quad \lim_{x \rightarrow \pi/2} f_p^s(x) = \frac{4}{\pi^2} \left(p - \sinh\left(\frac{\pi}{2}\right) \operatorname{csch}\left(\frac{\pi}{2p}\right) \right). \quad (35)$$

Thus, we have the following analogue of Theorem 2 for $\cosh x$ and $\sinh x$.

THEOREM 3. *Let $0 < x < \pi/2$. Then the two double inequalities*

$$\frac{4}{\pi^2} \left(1 - \cosh\left(\frac{\pi}{2}\right) \operatorname{sech}\left(\frac{\pi}{2p}\right) \right) < \frac{1 - \frac{\cosh x}{\cosh \frac{x}{p}}}{x^2} < \frac{1 - p^2}{2p^2} \quad (36)$$

and

$$\frac{4}{\pi^2} \left(p - \sinh\left(\frac{\pi}{2}\right) \operatorname{csch}\left(\frac{\pi}{2p}\right) \right) < \frac{p - \frac{\sinh x}{\sinh \frac{x}{p}}}{x^2} < \frac{1 - p^2}{6p} \quad (37)$$

hold for $p = 3, 4, 5, \dots$. In particular, the double inequality (36) is reversed when $p = 2$ while the double inequality (37) remains true when $p = 2$.

4. Application of the generalized D'Aurizio-Sándor inequalities to the Chebyshev polynomials of the second kinds

The first few Chebyshev polynomials of the second kind $U_n(x)$ ($n = 0, 1, 2, \dots$) are ([1, 3])

$$U_0(x) = 1, \quad (38)$$

$$U_1(x) = 2x, \quad (39)$$

$$U_2(x) = 4x^2 - 1, \quad (40)$$

$$U_3(x) = 8x^3 - 4x, \quad (41)$$

$$U_4(x) = 16x^4 - 12x^2 + 1, \quad (42)$$

$$U_5(x) = 32x^5 - 32x^3 + 6x, \quad (43)$$

$$U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1. \quad (44)$$

In this section, we apply Theorem 2 to $U_n(x)$ with $x = \cos \theta$. By means of the formula $U_n(\cos \theta) = \frac{\sin((n+1)\theta)}{\sin \theta}$, we obtain the following corollary.

COROLLARY 1. *Let $y \in (0, \frac{\pi}{2p})$. The double inequality*

$$\frac{p}{6} ((1 - p^2)y^2 + 6) < U_{p-1}(\cos y) < p - \frac{4}{\pi^2} \left(p - \csc \left(\frac{\pi}{2p} \right) \right) p y^2 \quad (45)$$

holds for $p = 2, 3, 4, 5, \dots$.

Proof. The double inequality (8) in Theorem 2 can be written as

$$p - \frac{p^2 - 1}{6p} x^2 < \frac{\sin x}{\sin \frac{x}{p}} < p - \frac{4}{\pi^2} \left(p - \csc \left(\frac{\pi}{2p} \right) \right) x^2, \quad x \in (0, \pi/2). \quad (46)$$

Letting $x/p = y$, we have

$$\frac{p}{6} ((1 - p^2)y^2 + 6) < \frac{\sin(py)}{\sin y} < p - \frac{4}{\pi^2} \left(p - \csc \left(\frac{\pi}{2p} \right) \right) p y^2, \quad y \in (0, \frac{\pi}{2p}). \quad (47)$$

Since $\frac{\sin(py)}{\sin y} = U_{p-1}(\cos y)$, the proof is completed. \square

EXAMPLE 1. Letting $p = 7$ in Corollary 1 results in the following inequality

$$7 - 56y^2 < 64 \cos^6 y - 80 \cos^4 y + 24 \cos^2 y - 1 < 7 - \frac{196(7 - \csc(\frac{\pi}{14}))}{\pi^2} y^2, \quad (48)$$

where $y \in (0, \frac{\pi}{14}) \approx (0, 0.2244)$ and $\frac{196(7 - \csc(\frac{\pi}{14}))}{\pi^2} \approx 49.7673$.

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